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# On the spin-up of an electrically conducting fluid Part 2. Hydromagnetic spin-up between infinite flat insulating plates

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The linear spin-up of a homogeneous electrically conducting fluid confined between infinite flat insulating plates is analyzed for the case in which a uniform magnetic field is applied normal to the boundaries. As in part 1 (Benton & Loper 1969), complete hydromagnetic interaction is embraced even within linearized equations. Approximate inversion of the exact Laplace transform solution reveals the presence of several flow structures: two thin Ekman–Hartmann boundary layers (one on each plate), which are quasi-steady on the time scale of spin-up, two thicker continuously growing magnetic diffusion regions, and an essentially inviscid, current-free core, which may or may not be present on the spin-up time scale, depending upon the growth rate of the magnetic diffusion regions. When a current-free core exists, it is found to spin-up at the same rate as the fluid within magnetic diffusion regions, although different physical mechanisms are at play. As a result, a single hydromagnetic spin-up time is derived, independently of the thickness of magnetic diffusion regions; this time is shorter than in the non-magnetic problem.

## 1. Introduction

As stated in part 1, we analyze the prototype linear hydromagnetic spin-up problem for an electrically conducting fluid, namely impulsively generated spin-up from one angular velocity $\Omega$ to the slightly different value $\Omega(1+\epsilon)$ in a 'container' formed by two infinite insulating disks. The imposed magnetic field, whose flow-induced distortion produces the electric currents necessary to complete the hydromagnetic coupling, is simply a uniform field $B_0$ perpendicular to the plates, which are situated at $z = \pm d$.

The notation, methodology, and results are direct, but interesting, extensions of those developed in part 1. Only the most important previous ideas are repeated here; self-containment of the present paper is deliberately sacrificed to conserve space. Frequent references to the earlier work will be necessary.

In §2 a heuristic discussion of the hydromagnetic spin-up is given together with a derivation of the spin-up time. This gives way, in § 3, to an exact mathematical formulation, and, in § 4, to the solution in the Laplace transform plane. Section 5 presents the approximate inversion, thereby confirming the conjectures of § 2. Results are summarized in § 6.

## 2. Preliminary discussion

The following results, derived in Gilman & Benton (1968) and in part 1, are indispensable for understanding the problem at hand.

(i) When a weak but steady difference in angular velocity is maintained between a single insulating flat plate and the electrically conducting fluid far from it, the steady flow field consists of two regions. An Ekman–Hartmann boundary layer exists to provide for the transition required of the angular velocity. Ekman suction (or blowing) velocity within this region, and also outside of it, is inhibited (compared to non-magnetic flow), because of the imposed axial magnetic field. Outside this boundary layer is a steady spatially uniform effectively inviscid 'magnetic diffusion region', which extends to axial infinity; an axial Hartmann electric current flows in it in the same sense as the axial Ekman velocity, having been induced by the shearing motion within the Ekman–Hartmann boundary layer.

(ii) The impulsively started, time-dependent approach to this steady state is characterized by three regions of flow (refer to figure 4 of part 1). A quasi-steady Ekman–Hartmann layer forms within a few boundary revolutions following the impulse. A much thicker magnetic diffusion region is called into existence to reduce the axial Hartmann current to zero at infinity by turning it into the radial direction; this region continually diffuses at the resistive rate, never becoming steady until it has reached infinite thickness. At finite times there is, beyond this region, a continuously shrinking current-free inviscid core, in which the Ekman velocity is larger than in non-magnetic flow (the enhancement being due to weak secondary motions, akin to Ekman pumping, induced within the magnetic diffusion region).

These results are next applied to the situation in which a second plate is present, at distance $2d$ from the original one; the fluid now spins up to the new angular velocity (if $\epsilon > 0$, which we take it to be for purposes of discussion). Relying on a result of the well-understood non-magnetic problem (Greenspan & Howard 1963), we adopt the working hypothesis (verified in § 5 below) that, at least for sufficiently early times, each boundary disturbs the flow in the same way as if it were acting in isolation; i.e. individual Ekman–Hartmann layers and magnetic diffusion regions are assumed to develop near each plate, exactly as in part 1. However, the existence of an external length scale in the two plate geometry (the plate separation, $2d$) implies that now the fluid dynamics can depend upon two parametric degrees of freedom not present in the isolated plate problem of part 1: namely, (*a*) the ratio of Ekman–Hartmann depth to total fluid depth, and (*b*) the ratio of the thickness of a magnetic diffusion region to total fluid depth. Each Ekman–Hartmann layer quickly reaches a quasi-steady thickness of order $\beta^{-1}(\nu/\Omega)^{\frac{1}{2}}$ (cf. part 1, (55)–(59)) so the first ratio is $(2\beta)^{-1}E^{\frac{1}{2}}$

where the Ekman number $E$ is $\nu/d^2\Omega$. Because $\beta \geqslant 1$ (part 1, (52)) and the physically interesting situation requires $E \ll 1$, we need only consider the case in which Ekman–Hartmann layers occupy a negligible fraction of the total fluid depth. For the second ratio, attention is restricted to the moderate range of field strengths $\alpha \leqslant O(1)$, so that the Alfvén mode is not strongly present. Then unsteady magnetic diffusion regions are of thickness $2(\lambda t)^{\frac{1}{2}}$ (see part 1 following (79)), so their proportion of the total fluid depth is of order $(E\tau/\delta)^{\frac{1}{2}}$, where $\delta$ is the magnetic Prandtl number $\nu/\lambda$, and $\tau = \Omega t$. Since $\delta$ as well as $E$ is typically much smaller than one, no definite bounds can be placed on this ratio, and we must expect the possibility that spin-up depends strongly on the relative thickness of magnetic diffusion regions.

Consider first the situation where $E\tau_s \ll \delta \ll 1$ ($\tau_s$ is the, as yet undetermined, non-dimensional spin-up time), so that the magnetic diffusion regions, as well as the Ekman–Hartmann layers, remain thin throughout spin-up (see figure 1). The bulk of the fluid is then an inviscid, current-free core, which must accordingly spin-up by exactly the same mechanism as in the non-magnetic problem: conservation of angular momentum for fluid rings driven radially inward by Ekman secondary flow. In the present case, the driving secondary flow in the inviscid core is stronger, by a factor $\beta$, than in the non-magnetic case, because the Ekman suction at the outer edge of the magnetic diffusion region is so enhanced (part 1, (84)). Consequently, the spin-up time is estimated by conventional methods (e.g. Greenspan 1968, §2.4) to be

$$\tau_s \equiv \Omega t_s = \beta^{-1}E^{-\frac{1}{2}}, \tag{1}$$

and the inequality above is satisfied if $E^{\frac{1}{2}} \ll \delta \ll 1$. The non-dimensional azimuthal velocity function (refer to part 1, (6)) is in this case expected to be

$$V(\tau) = 1-\exp(-\beta E^{\frac{1}{2}}\tau). \tag{2}$$

For non-magnetic flow $\alpha = 0$, $\beta = 1$ so $\tau_s = E^{-\frac{1}{2}}$, as in Greenspan & Howard's problem. Spin-up of an inviscid current-free core by hydromagnetically enhanced Ekman secondary flow is predicted to be never slower than the classical spin-up.

If, on the other hand, $\delta \ll E\tau_s \ll 1$, then the magnetic diffusion regions rather quickly (i.e. for $\tau \sim \delta/E \ll \tau_s$) reach thicknesses comparable with $d$; in a linear problem such as this, it may be supposed that in their subsequent diffusion they overlap or inter-penetrate each other but do not undergo any strong amplifying nonlinear interaction (see figure 2). (It is important to recall that, according to part 1, the perturbations induced within such regions are weak, of order $\delta^{\frac{1}{2}} \ll 1$.) Thus, the picture emerges of two thin quasi-steady Ekman–Hartmann layers separated by an inviscid but resistive, current-carrying hydromagnetic core, within which spin-up takes place by the joint action of Ekman secondary flow (which is *weaker* than in the non-magnetic case; refer to part 1, (69) and figure 1), and an electromagnetic body torque arising from a Hartmann current system in meridional planes (figure 2; part 1, (69) and figure 2). Mathematically this situation is described by the inviscid version of the tangential momentum equation combined with conservation of mass (part 1, (12), (16)):

$$V_\tau = (W+2\alpha^2 B)_\zeta. \tag{3}$$

Recognizing that $B$ measures the Hartmann current (part 1, (10)), that it is in the same sense as the axial Ekman flow (Gilman & Benton 1968), and that it obeys the same symmetry restrictions (and, in this case, boundary conditions) as $W$, we estimate the spin-up time in terms of $W + 2\alpha^2 B$ evaluated at the outer edge of the very thin quasi-steady Ekman–Hartmann layers. For the case at

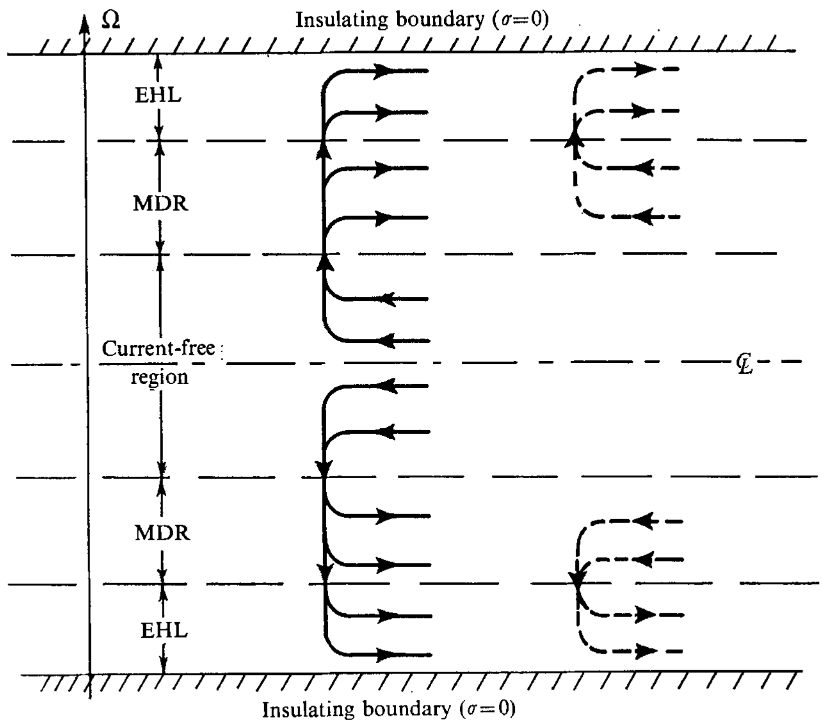


FIGURE 1. Magnetic diffusion region thin during spin-up: —, schematic meridional topology of lines tangent to velocity field; – –, electric current; EHL, Ekman–Hartmann layer; MDR, magnetic diffusion region.

hand, where the magnetic diffusion regions are much thicker than the Ekman–Hartmann layers, the appropriate expression is obtained by letting $r \to \infty$ in (69) of part 1, which gives

$$W_1 + 2iB_1 = -\frac{2i}{\beta + i\gamma} = -\frac{2(\gamma + i\beta)}{\beta^2 + \gamma^2}. \quad (4)$$

Consequently,

$$W_1 + 2\alpha^2 B_1 = -\frac{2}{\beta^2 + \gamma^2}(\gamma + \alpha^2\beta) = -\beta, \quad (5)$$

where use has been made of the definitions of $\beta$, $\gamma$ (part 1, (52)). Since this value is maintained at the edge of thin, quasi-steady Ekman–Hartmann layers (i.e. at $z \doteq \pm d$, which is $\zeta \doteq \pm E^{-\frac{1}{2}}$ in the scaled variables of part 1), the estimated dimensionless spin-up time is again $\beta^{-1}E^{-\frac{1}{2}}$. Apparently, the reduction in Ekman suction from the classical value due to inhibited Ekman pumping in the Ekman–Hartmann layers is exactly compensated by the accelerating electromagnetic body torque arising from the Hartmann electric current system. The conclusion is that, if our approximations are well-founded, then spin-up proceeds at the same rate whether the magnetic diffusion regions are very thin or very thick.

The same reasoning applied to the intermediate case where these regions are thick compared to the Ekman–Hartmann layers, but not comparable in thickness with the total fluid depth, suggests that fluid both inside and outside the

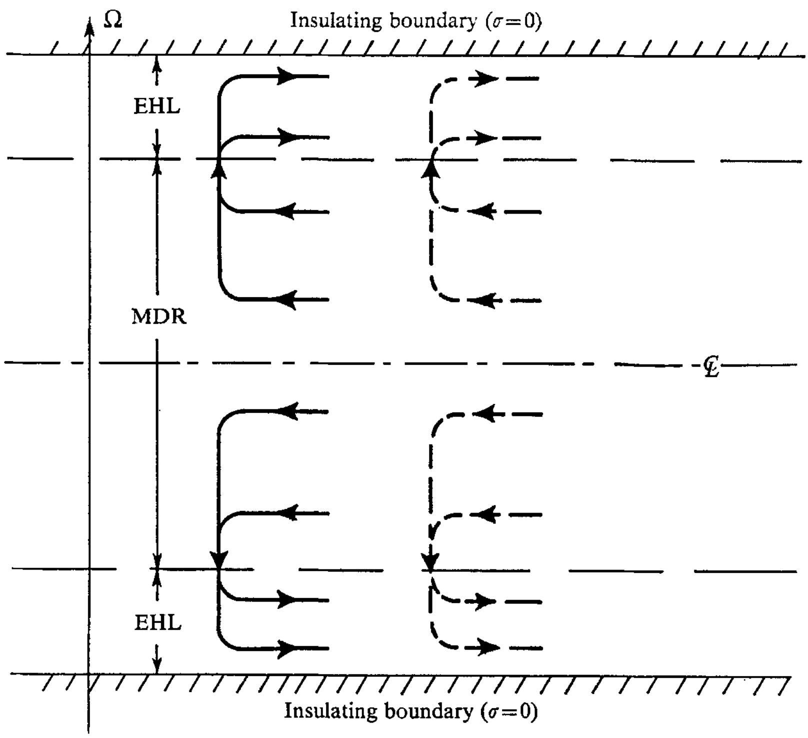


FIGURE 2. Magnetic diffusion region thick during spin-up: —, schematic meridional topology of lines tangent to velocity field; – –, electric current; EHL, Ekman–Hartmann layer; MDR, magnetic diffusion region.

magnetic diffusion regions spins-up in unison (but because of distinctly different physical mechanisms). In effect, a Taylor–Proudman constraint appears to be operating, so that the angular velocity remains independent of depth within and outside the magnetic diffusion regions. We therefore anticipate no dependence of spin-up time on the ratio $(E/\delta)$, and this result is confirmed by the more rigorous analysis which follows.

## 3. Mathematical formulation

For impulsively started hydromagnetic spin-up in the two-plate geometry, the fundamental equations are still (1)–(3) of part 1, but the initial and boundary conditions now take the form:

$$\text{at } t = 0, \quad \mathbf{v} = r\Omega\hat{\theta}, \quad \mathbf{B} = B_0\hat{z}, \tag{6}$$

$$\text{for } t > 0, \quad \text{at} \quad z = \pm d, \quad \mathbf{v} = r\Omega(1+\epsilon)\hat{\theta}, \quad \mathbf{B} \text{ continuous.} \tag{7}$$

The occurrence of the external length $d$ in the present problem makes certain changes in scaling desirable. For example, $z$ is scaled with $d$ rather than Ekman depth, in order to remove parameters from the boundary conditions. The previous scaling for the horizontal components of both $\mathbf{v}$ and $\mathbf{B}$ (part 1, (6), (8)) is retained, so that the corresponding non-dimensional functions are still expected to be independent of kinematic viscosity outside of Ekman–Hartmann layers (i.e. of order 1 or less with respect to $E$, $\delta$). Also, an added pressure function is now necessary, because the fluid depth is finite. Equations analogous to (6), (7), (8), (10) of part 1 are

$$\mathbf{v}(r,z,t) = r\Omega\hat{\theta} + \Omega\epsilon[rU(\zeta,\tau)\,\hat{r} + rV(\zeta,\tau)\,\hat{\theta} + dW(\zeta,\tau)\,\hat{z}], \tag{8}$$

$$\Pi(r,z,t) = \tfrac{1}{2}r^2\Omega^2 + \tfrac{1}{2}\Omega^2\epsilon[r^2P(\tau) + 2d^2Q(\zeta,\tau)], \tag{9}$$

$$\mathbf{B}(r,z,t) = B_0\hat{z} + B_0\mu\sigma(\nu\Omega)^{\frac{1}{2}}\epsilon[rA(\zeta,\tau)\,\hat{r} + rB(\zeta,\tau)\,\hat{\theta} + dC(\zeta,\tau)\,\hat{z}], \tag{10}$$

$$\mathbf{j}(r,z,t) \equiv \mu^{-1}\nabla\times\mathbf{B} = B_0\sigma(\nu\Omega)^{\frac{1}{2}}d^{-1}\epsilon\left[-r\frac{\partial B}{\partial\zeta}\hat{r} + r\frac{\partial A}{\partial\zeta}\hat{\theta} + 2dB\hat{z}\right], \tag{11}$$

where $\zeta = z/d$, $\tau = \Omega t$.

The linearized, unsteady perturbation problem now forms a three-parameter set:

$$U_\tau - EU_{\zeta\zeta} = 2V + 2\alpha^2E^{\frac{1}{2}}A_\zeta - P, \tag{12}$$

$$V_\tau - EV_{\zeta\zeta} = -2U + 2\alpha^2E^{\frac{1}{2}}B_\zeta, \tag{13}$$

$$\delta A_\tau - EA_{\zeta\zeta} = E^{\frac{1}{2}}U_\zeta, \tag{14}$$

$$\delta B_\tau - EB_{\zeta\zeta} = E^{\frac{1}{2}}V_\zeta, \tag{15}$$

$$\delta C_\tau - EC_{\zeta\zeta} = E^{\frac{1}{2}}W_\zeta, \tag{16}$$

$$W_\zeta = -2U, \tag{17}$$

$$C_\zeta = -2A, \tag{18}$$

$$Q_\zeta = EW_{\zeta\zeta} - W_\tau. \tag{19}$$

The three dimensionless parameters are given by

$$\alpha = \left(\frac{\sigma}{2\rho\Omega}\right)^{\frac{1}{2}} B_0 = \text{magnetic interaction parameter}, \tag{20}$$

$$\delta = \nu/\lambda = \sigma\mu\nu = \text{magnetic Prandtl number}, \tag{21}$$

$$E = \nu/d^2\Omega = \text{Ekman number}. \tag{22}$$

For the physical situations of interest, both $E$ and $\delta$ are much smaller than one. In part 1 it was seen that the transition from weak to strong magnetic effects occurs for $\alpha$ of order 1, so in the present problem we will ultimately deal only with $\alpha \leqslant O(1)$.

The initial and boundary conditions for the system (12)–(19) are:

$$\text{at } \tau = 0, \quad U = V = W = P = Q = A = B = C = 0, \tag{23}$$

$$\text{for } \tau > 0, \quad \text{at } \zeta = \pm 1, \quad V = 1, \quad U = W = A = B = C = 0, \tag{24}$$

where (as in part 1) the perturbation magnetic field vanishes throughout the insulating half-spaces and therefore vanishes at $\zeta = \pm 1$ by continuity of the field.

In terms of the complex quantities,

$$F(\zeta,\tau) = U(\zeta,\tau)+iV(\zeta,\tau), \quad (25)$$

$$M(\zeta,\tau) = A(\zeta,\tau)+iB(\zeta,\tau), \quad (26)$$

which contain the main functions of interest, the problem is compactly expressed as

$$F_\tau - EF_{\zeta\zeta} + 2iF = 2\alpha^2 E^{\frac{1}{2}} M_\zeta - P(\tau), \quad (27)$$

$$\delta M_\tau - EM_{\zeta\zeta} = E^{\frac{1}{2}} F_\zeta, \quad (28)$$

$$F(\pm 1,\tau) = i, \quad F(\zeta,0) = M(\zeta,0) = M(\pm 1,\tau) = \mathrm{Re}\int_{-1}^{1} F\,d\zeta = 0. \quad (29)$$

It is readily verified that Greenspan & Howard's non-magnetic problem is recovered from (27) in the limit $\alpha \to 0$.

It is useful at this point to prove that, as predicted above, the fluid outside of the Ekman–Hartmann boundary layers does in fact spin-up in the columnar fashion consistent with the Taylor–Proudman theorem, regardless of the thickness of magnetic diffusion regions. This follows from an examination of (27) or (28) on suitably stretched spatial and temporal scales. For the restricted range of interest for $\alpha$ ($\leqslant O(1)$), $\beta$ is of order 1, so the Ekman–Hartmann layers are of order $E^{\frac{1}{2}}$ in thickness. To represent spatial variations on scales greater than that, $\zeta$ is stretched by setting $\zeta = E^{\frac{1}{2}-a}\zeta^*$, where $\zeta^*$ is of order 1 and $0 < a \leqslant \frac{1}{2}$ (because $a = 0$ gives the scale of an Ekman layer and $a = \frac{1}{2}$, that of the plate separation). The appropriate time scale is that for spin-up, which, for $\beta = O(1)$, is anticipated to be of order $E^{-\frac{1}{2}}$ (short-period oscillations and initial boundary-layer development are thereby filtered out of this analysis). So let $\tau = E^{-\frac{1}{2}}\tau^*$ with $\tau^* = O(1)$. The basic functions, $F, M, P$ are properly scaled to be of order 1 or less within the core (as can be partially seen from (27), (29), which together suggest that $F, P$ rise from their initial values of 0 to final values of $i$ and 2, respectively, during spin-up). In the stretched co-ordinates, (27), (28) become

$$E^{\frac{1}{2}}F_{\tau^*} - E^{2a}F_{\zeta^*\zeta^*} + 2iF = 2\alpha^2 E^a M_{\zeta^*} - P(\tau^*),$$

$$E^{\frac{1}{2}-a}\delta M_{\tau^*} - E^a M_{\zeta^*\zeta^*} = F_{\zeta^*}.$$

Clearly, for any $a > 0$, the dominant momentum balance in an asymptotic sense ($E \to 0$) is geostrophic:

$$2iF \equiv -2V + 2iU = -P(\tau^*),$$

which implies not only that the order 1 azimuthal flow obeys the Taylor–Proudman constraint (as is seen also from the scaled induction equation),

$$\partial V/\partial\zeta^* = 0,$$

but also that the radial motion $U$ is zero to leading order (as in the non-magnetic problem). Perturbation velocity components of higher order do not satisfy such a constraint and can vary appreciably within magnetic diffusion regions. However, the diffusive growth and possible interpenetration of magnetic diffusion regions is too weak an effect to disturb the basic geostrophic balance which exists outside the Ekman–Hartmann layers, so the dominant order 1 process, spin-up, is unaffected by the thickness of magnetic diffusion regions.

## 4. Exact Laplace transform solution

Use of the same Laplace transform notation as in part 1 (28), leads to the problem,

$$\left.\begin{aligned} E\bar{F}'' - (s+2i)\bar{F} &= -2\alpha^2 E^{\frac{1}{2}}\bar{M}' + \bar{P}(s),\\ E\bar{M}'' - \delta s\bar{M} &= -E^{\frac{1}{2}}\bar{F}',\\ \bar{F}(\pm 1, s) = is^{-1}, \quad \bar{M}(\pm 1, s) &= \mathrm{Re}\int_{-1}^{1}\bar{F}(\zeta, s)\,d\zeta = 0, \end{aligned}\right\} \tag{30}$$

where primes denote differentiation with respect to $\zeta$. The exact solution is given by

$$\begin{aligned}\bar{F}(\zeta, s) &= \frac{i(s-2i)}{sD}[(\tilde{k}^2-\tilde{m}^2)\theta - (k^2-m^2)\tilde{\theta}]\\ &\quad + \frac{2i}{D}[E^{-\frac{1}{2}}\tilde{\theta} - (\tilde{k}^2-\tilde{m}^2)]\left[k(s+2i-m^2)\frac{\cosh E^{-\frac{1}{2}}k\zeta}{\sinh E^{-\frac{1}{2}}k} - m(s+2i-k^2)\frac{\cosh E^{-\frac{1}{2}}m\zeta}{\sinh E^{-\frac{1}{2}}m}\right],\end{aligned} \tag{31}$$

$$\bar{M}(\zeta, s) = -\frac{2i}{D}(s+2i)[E^{-\frac{1}{2}}\tilde{\theta} - (\tilde{k}^2-\tilde{m}^2)]\left[\frac{\sinh E^{-\frac{1}{2}}k\zeta}{\sinh E^{-\frac{1}{2}}k} - \frac{\sinh E^{-\frac{1}{2}}m\zeta}{\sinh E^{-\frac{1}{2}}m}\right], \tag{32}$$

$$\bar{P}(s) = -\frac{i(s^2+4)}{sD}[(\tilde{k}^2-\tilde{m}^2)\theta - (k^2-m^2)\tilde{\theta}]. \tag{33}$$

In these expressions, a tilda indicates a complex conjugate with $s$ regarded as real, and

$$\theta = k(s+2i-m^2)\coth E^{-\frac{1}{2}}k - m(s+2i-k^2)\coth E^{-\frac{1}{2}}m, \tag{34}$$

$$D = 2E^{-\frac{1}{2}}s\theta\tilde{\theta} - (s+2i)(\tilde{k}^2-\tilde{m}^2)\theta - (s-2i)(k^2-m^2)\tilde{\theta}, \tag{35}$$

$$k = [n + (n^2-q^2)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad \mathrm{Re}\,k \geqslant 0, \tag{36}$$

$$m = [n - (n^2-q^2)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad \mathrm{Re}\,m \geqslant 0, \tag{37}$$

$$n = \tfrac{1}{2}[(1+\delta)s + 2\alpha^2 + 2i], \tag{38}$$

$$q = (\delta s)^{\frac{1}{2}}(s+2i)^{\frac{1}{2}}, \quad \mathrm{Re}\,q \geqslant 0. \tag{39}$$

A comparison of (31)–(39) above with (30)–(35) of part 1 reveals both similarities and differences. The quantities $k, m, n, q$ are identical. The factor

$$[(s+2i)^{\frac{1}{2}} + (\delta s)^{\frac{1}{2}}],$$

which gives rise to branch points at $s = 0, -2i$ in the semi-infinite problem, is now absent, but instead we have the denominator function $D(s)$. The horizontal velocity function now has a part independent of $\zeta$ which clearly satisfies the Taylor–Proudman theorem and which is closely related to the pressure function $P(\tau)$. The spatial dependence is, apart from differences in scaling, again of the same exponential type, but both positive and negative exponents occur because of the reflexional symmetry about $\zeta = 0$. In part 1, there are no terms comparable with $(\sinh E^{-\frac{1}{2}}k)^{-1}$, $(\sinh E^{-\frac{1}{2}}m)^{-1}$, and this is a significant new feature.

A systematic search for the location and type of singularities of these exact transform solutions reveals several important facts. In particular, many incipiently singular points in the $s$-plane are, in fact, regular points of the solution functions. Consider, for example, the branch points of the functions $k$, $m$ at the two values of $s$ for which $n^2 = q^2$. In crossing the associated branch cuts, $k$ and $m$

become interchanged but that transformation leaves $\bar{F}$, $\bar{P}$, $\bar{M}$ unaffected. Also, since $\theta$, $D$, $\bar{F}$, $\bar{P}$, $\bar{M}$ are even functions of $k$ and $m$, there are no branch points at the zeros of $k$ or $m$. Thus, as in Greenspan & Howard, the solutions appear to possess no branch points whatever. Furthermore, the functions $\bar{F}$, $\bar{P}$, $\bar{M}$ remain bounded at values of $s$, for which $k = 0$, $m = 0$, $\sinh E^{-\frac{1}{2}}k = 0$, $\sinh E^{-\frac{1}{2}}m = 0$, $\theta = 0$, so there are no poles at such locations. Of course, the main function of interest, $\bar{F}$, does have an isolated simple pole at $s = 0$, whose residue yields the steady state solution. All other poles arise from the zeros of $D$, and are discussed more fully in § 5.

It is not trivial, but none the less straightforward, to take the non-magnetic limit ($\alpha \to 0$) of (31) and recover the solution obtained by Greenspan & Howard. Also, the steady-state hydromagnetic solution, found as in part 1, is

$$F(\zeta, \infty) = \lim_{s\to 0} s\bar{F}(\zeta, s) = i, \quad P(\infty) = 2, \quad M(\zeta, \infty) = 0. \tag{40}$$

Here it should be noted that this steady state (or long time) behaviour of $F$ arises from the space-independent term; the space-dependent parts, which give the boundary layers and magnetic diffusion regions, ultimately decay to zero.

## 5. Approximate Laplace inversion

We turn now to a study of the transform functions in the physically meaningful range of parameter space given by $0 < E \ll 1$, $0 \leqslant \delta \ll 1$, $\alpha \leqslant O(1)$. The basic approximation procedure is closely related to that which was successful in part 1; knowledge of the location of singularities is utilized in choosing contours for the inversion integrals, along which simplifying assumptions can be made about the various terms that make up the exact solutions. Our main goal is to understand the gross behaviour of $F$ for both early times and on the time-scale of spin-up.

The first stage of approximation involves a Taylor expansion of the functions $k$, $m$ in powers of $\delta^{\frac{1}{2}}$ valid for $\delta \ll 1$. Since $k$, $m$ are identical to those of part 1, the expansions are those given in (72, 73) of that paper, and are valid for the same wide range of conditions explained there. To dominant order in $\delta^{\frac{1}{2}}$,

$$\left.\begin{aligned} k \doteq k_0 &= (s+2\alpha^2+2i)^{\frac{1}{2}}, & \tilde{k} \doteq \tilde{k}_0 &= (s+2\alpha^2-2i)^{\frac{1}{2}}, \\ m \doteq m_0 &= \left[\frac{\delta s(s+2i)}{s+2\alpha^2+2i}\right]^{\frac{1}{2}}, & \tilde{m} \doteq \tilde{m}_0 &= \left[\frac{\delta s(s-2i)}{s+2\alpha^2-2i}\right]^{\frac{1}{2}}. \end{aligned}\right\} \tag{41}$$

With these expressions, it is readily verified that $\bar{F}$ remains finite in each of the following limits:

$$\begin{gathered} k_0 \to 0, \quad \tilde{k}_0 \to 0, \quad m_0 \to 0, \quad \tilde{m}_0 \to 0, \quad m_0 \to \infty, \quad \tilde{m}_0 \to \infty, \\ \theta \to 0, \quad \tilde{\theta} \to 0, \quad \theta \to \infty, \quad \tilde{\theta} \to \infty, \quad \sinh E^{-\frac{1}{2}}k_0 \to 0, \\ \sinh E^{-\frac{1}{2}}\tilde{k}_0 \to 0, \quad \sinh E^{-\frac{1}{2}}m_0 \to 0, \quad \sinh E^{-\frac{1}{2}}\tilde{m}_0 \to 0. \end{gathered}$$

The only singularities therefore, are still poles at $s = 0$ and the zeros of $D$.

A second approximation, valid for $\tau \leqslant O(1)$, is now introduced in order to verify the hypothesis of § 2: that, for early times, the flow evolves as the sum of flows due to two boundaries acting in isolation (rapidly propagating waves with suitable reflexion properties could vitiate such a conclusion, if present). For times $\tau$ not exceeding order 1, the fundamental inversion contour (a vertical straight line anywhere in the right half $s$-plane) can be deformed, so that it indents into the right half-plane only along the segment between the points $s = 1 \pm 3i$, thereby avoiding any singularities along the imaginary axis (such as at $s = 0, \pm 2i$) by an amount no less than order 1. We suppose that the other singularities (the zeros of $D$) are so distributed in the left half-plane, that the contour need never approach more closely than order 1 to any of them. Then, everywhere on the deformed contour, $|k_0| \geqslant 1$, $|m_0| \leqslant |k_0|$, so the following approximations are valid:

$$\coth E^{-\frac{1}{2}}k \doteq 1, \quad \sinh E^{-\frac{1}{2}}k \doteq \tfrac{1}{2}\exp(E^{-\frac{1}{2}}k_0), \quad \cosh E^{-\frac{1}{2}}k\zeta \doteq \tfrac{1}{2}\exp(E^{-\frac{1}{2}}k_0|\zeta|),$$
$$s+2i-m^2 \doteq s+2i, \quad \theta \doteq (s+2i)\,k_0,$$
$$E^{-\frac{1}{2}}\tilde{\theta}-(\tilde{k}^2-\tilde{m}^2) \doteq E^{-\frac{1}{2}}\tilde{\theta}, \quad D \doteq 2E^{-\frac{1}{2}}s\theta\tilde{\theta}.$$

Introduction of these approximations and (41) into the expression for $\bar{F}$ leads to

$$\bar{F}(\zeta, s) \doteq \frac{iE^{\frac{1}{2}}}{2s^2(s+2i)}[(s+2i)\,\tilde{k}_0-(s-2i)\,k_0]$$
$$+\frac{i}{s}\left[\exp[-E^{-\frac{1}{2}}k_0(1-|\zeta|)]+\frac{2\alpha^2 m_0}{(s+2i)\,k_0}\frac{\cosh E^{-\frac{1}{2}}m_0\zeta}{\sinh E^{-\frac{1}{2}}m_0}\right]. \qquad (42)$$

The branch points which now exist at $s = -2\alpha^2 \pm 2i$ arise because the approximations above do not always preserve the evenness or oddness of a function with respect to $k$. Effectively, infinite sequences of poles describing damped inertial oscillations are replaced by branch cuts (as in Greenspan & Howard).

The term independent of $\zeta$ in (42) is of order $E^{\frac{1}{2}}$ compared to the others and is therefore negligible (reflecting the fact that the core spins up by only an insignificant amount during the first radian or so of rotation following the impulse). The exponential term involving $k_0$ describes the development of thin Ekman–Hartmann layers on the two boundaries, and is equivalent to the one-plate solution under these same approximations (see part 1, (61), (74)). Since

$$E^{-\frac{1}{2}}m_0 = \left(\frac{\delta}{E}\right)^{\frac{1}{2}}\left[\frac{s(s+2i)}{s+2\alpha^2+2i}\right]^{\frac{1}{2}},$$

the exponential terms in $m_0$ describing magnetic diffusion regions reduce to the isolated plate results (part 1, (74)) only if $E \leqslant \delta$, but this is just equivalent to the requirement that a layer, whose thickness growth-rate is $2(\lambda t)^{\frac{1}{2}}$ (i.e. a magnetic diffusion region), be thin compared to the plate separation, $2d$, after an elapsed time of order $\Omega^{-1}$ (or $\tau \sim 1$). In other words, when magnetic diffusion regions are thin compared to the total depth of fluid, then they too develop as they would in the absence of the second boundary. Their possible interpenetration (which requires sufficiently small $\delta$) is accounted for largely by the factor $(\sinh E^{-\frac{1}{2}}m_0)^{-1}$. Even when the two exponentials in $\sinh E^{-\frac{1}{2}}m_0$ overlap, as for example when they are each comparable so that this factor cannot be approximated by $\frac{1}{2}\exp(E^{-\frac{1}{2}}m_0)$, but rather by $E^{-\frac{1}{2}}m_0$, then the ratio of the coefficient of the $m$-layer exponentials to those of the $k$-layer exponentials is

$$2\alpha^2 m_0/(s+2i)\,k_0 \sinh E^{-\frac{1}{2}}m_0 \doteq 2\alpha^2 E^{\frac{1}{2}}/(s+2i)\,k_0,$$

and this is, in magnitude, much smaller than 1 everywhere on the contour. Hence, radial and tangential velocities within magnetic diffusion regions remain weak compared to those in Ekman–Hartmann layers even after the magnetic regions interact with each other.

Attention is now directed to an examination of the function $F$ on the anticipated time-scale of spin-up, which for $\alpha \leqslant O(1)$ is $E^{-\frac{1}{2}}$, as in the non-magnetic problem. The same expressions for $k$, $m$ as in (41) are appropriate, but the inversion contour must now run much closer to the important singularities at $s = 0, \pm 2i$; in order that $\exp(s\tau)$ in the inversion integrals be of bounded magnitude along the contour, it should indent into the right half-plane by an amount no greater than order $E^{\frac{1}{2}}$. An approximation is sought uniformly valid through times of order $E^{-\frac{1}{2}}$ only. On this long time scale, the dominant contributions to the inversion integrals arise from portions of the contour closest to the imaginary axis. In fact, as in Greenspan & Howard, the singularities within a distance of order $E^{\frac{1}{2}}$ from the origin are expected to be most important, so we confine attention to the transform functions in such a neighbourhood of $s = 0$ by setting

$$s = E^{\frac{1}{2}}s_1, \quad \text{where} \quad |s_1| \leqslant O(1). \tag{43}$$

This effectively filters out of the subsequent analysis all inertial oscillations (which arise from the vicinity of $s = \pm 2i$), and can be rationalized heuristically as follows. Greenspan & Howard have shown conclusively that both individually and collectively such oscillations are unimportant for describing non-magnetic spin-up. Furthermore, part 1 (following (67)) demonstrates that ordinarily these oscillations damp out even more rapidly when a magnetic field is present. A possible exception could be the highly persistent hydromagnetically driven inertial oscillations discussed in § 5 of part 1, but such strong vertical motions as those are greatly suppressed in the present problem because of the symmetry imposed by the second boundary. The only other type of oscillatory motions to be expected here, on physical grounds, are Alfvén modes, but they cannot be strongly present, because, when $\alpha \leqslant O(1)$, the field is relatively weak or the distorting effect of rotation is rather strong. A more appropriate remark is that in spite of their assumed unimportance from our admittedly limited viewpoint of spin-up, these oscillations certainly deserve further study for their inherent interest. However, linearity of the present problem suggests that the necessary residue calculation implied in such a study will provide terms to be *added on* to the results found below, but it is doubtful whether spin-up itself would be seriously affected. The interested reader can find this statement verified in a report by Loper & Benton (1970), where further details are given.

The main task now is to show that spin-up does proceed at the same rate regardless of the relative magnitudes of $E$ and $\delta$, so long as both are much smaller than one. Introduction of (43) into (41), and neglect of terms of order $E^{\frac{1}{2}}$ or smaller compared to those retained, leads to

$$k_0 \doteq \beta + i\gamma, \tag{44}$$

$$m_0 \doteq (1 - i\alpha^2)^{-\frac{1}{2}}\, \delta^{\frac{1}{2}} E^{\frac{1}{4}} s_1^{\frac{1}{2}}, \tag{45}$$

$$\theta \doteq 2ik_0 + 2\alpha^2 m_0 \coth E^{-\frac{1}{2}} m_0. \tag{46}$$

Equations (44), (45) show that for all $\alpha \leqslant O(1)$, $|s_1| \leqslant O(1)$, $|k_0|$ is an order 1 constant independent of $s_1$, whereas $|m_0|$ never exceeds of order $\delta^{\frac{1}{2}}E^{\frac{1}{4}}$, which is much less than 1; so in the range of interest $|m_0| \ll |k_0|$.

Further progress rests on a demonstration that the second term of $\theta$ in (46) is always unimportant compared to the first. It clearly vanishes as $\alpha \to 0$; as either $\delta$ or $s_1$ tend to 0, it approaches $2\alpha^2 E^{\frac{1}{2}}$, which is negligible compared to $2ik_0$. Indeed, only at values of $s_1$ such that $|\coth E^{-\frac{1}{2}}m_0| \gg 1$, could the term in question conceivably be important. Now since

$$|\coth(x+iy)| = \left(\frac{\cosh 2x + \cos 2y}{\cosh 2x - \cos 2y}\right)^{\frac{1}{2}},$$

$|\coth(x+iy)|$ has large values only near the imaginary axis, dropping rapidly to order one in magnitude everywhere once $|x|$ exceeds 1 or so. In terms of $s_1$, $\coth E^{-\frac{1}{2}}m_0$ is infinite only at the isolated values given by

$$s_1 = -N^2\pi^2(1-i\alpha^2)E^{\frac{1}{2}}\delta^{-1}, \tag{47}$$

where $N$ is an integer. Since these points lie along a ray in the second quadrant of the $s_1$-plane which makes an angle $\tan^{-1}\alpha^2$ with the negative real axis, the path of integration (which is in the right half-plane along $\operatorname{Re} s_1 = O(1)$) avoids the places where $|\coth E^{-\frac{1}{2}}m_0| \gg 1$. Furthermore, the resulting infinities of $\theta$ for these values of $s_1$ are regular points of $\bar{F}$. It will soon be seen that the constant, order 1 part of $\theta$, due to the term $2ik_0$ is in fact what produces a simple zero of $D$. The second term of $\theta$ is incapable of contributing significantly to spin-up, because of the topology of the complex coth function and the location of the contour. (In Loper & Benton (1970), it is shown that a discrete set of poles of $\bar{F}$ does exist, because $D = 0$ at values of $s_1$ *close* to those in (47); however, the residue calculation given there reveals only weak velocities compared to the spin-up mode.) With the approximation,

$$\theta \doteq 2ik_0 \doteq 2i(\beta + i\gamma), \tag{48}$$

it is easily seen that $D$ reduces to

$$D \doteq 8(\beta^2+\gamma^2)(s_1+\beta) = 8(\beta^2+\gamma^2)E^{-\frac{1}{2}}(s+\beta E^{\frac{1}{2}}). \tag{49}$$

Since no reference needed to be made in the preceding argument to the relative ordering of $E$, $\delta$, (49) shows that $D$ has a simple zero at the expected location of the spin-up pole, regardless of the thickness of magnetic diffusion regions. The final approximate transform function, valid on the time scale of order $E^{-\frac{1}{2}}$, and with initial boundary-layer development and short period oscillations omitted, is now obtained by introducing (43), (44), (45), (48), (49) into (31), and ignoring terms small compared to those retained:

$$\bar{F}(\zeta, s) \doteq \bar{F}_1(s) + \bar{F}_2(\zeta, s) + \bar{F}_3(\zeta, s), \tag{50}$$

where

$$\bar{F}_1(s) = \frac{i\beta E^{\frac{1}{2}}}{s(s+\beta E^{\frac{1}{2}})} = \frac{i}{s} - \frac{i}{s+\beta E^{\frac{1}{2}}}, \tag{51}$$

$$\bar{F}_2(\zeta, s) = \frac{i}{s+\beta E^{\frac{1}{2}}}\exp\left[-E^{-\frac{1}{2}}(\beta+i\gamma)(1-|\zeta|)\right] \tag{52}$$

$$\bar{F}_3(\zeta, s) = \frac{\alpha^2 m_0}{(\beta+i\gamma)(s+\beta E^{\frac{1}{2}})}\,\frac{\cosh E^{-\frac{1}{2}}m_0\zeta}{\sinh E^{-\frac{1}{2}}m_0}. \tag{53}$$

The inversion of (51) gives the dominant spin-up behaviour of the fluid outside of Ekman–Hartmann layers as anticipated in § 2:

$$F_1(\tau) = i(1-\exp(-\beta E^{\frac{1}{2}}\tau)). \tag{54}$$

The imaginary part of (54), representing the inviscid azimuthal velocity, is plotted *versus* time in figure 3 to demonstrate the more rapid fluid response with increasing magnetic interaction parameter $\alpha$. The inversion of (52) clearly shows that quasi-steady Ekman–Hartmann layers on each boundary, like that of part 1 (e.g. (53)), slowly decay as the fluid spins up:

$$F_2(\zeta,\tau) = i\exp[-E^{-\frac{1}{2}}(\beta+i\gamma)(1-|\zeta|)-\beta E^{\frac{1}{2}}\tau]. \tag{55}$$

Although (53) can be inverted, with $m_0$ as given in (45), it would be misleading to do so, because we have not proven that (53) is a valid approximation for the

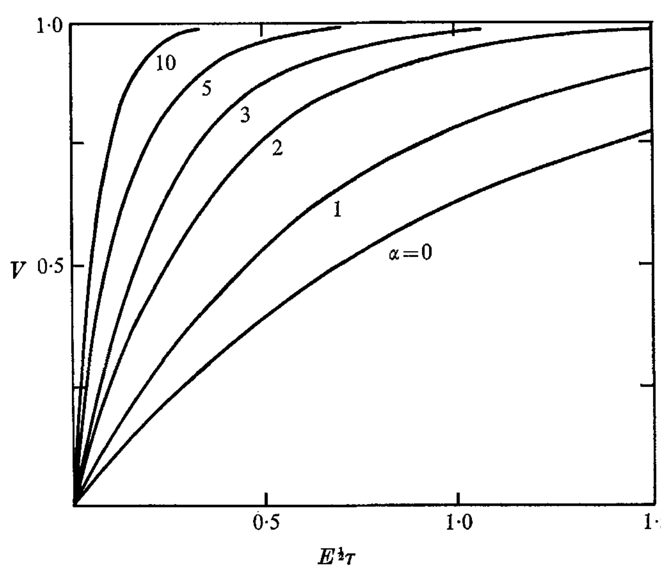


FIGURE 3. Inviscid azimuthal perturbation velocity $V$, *versus* spin-up time $E^{\frac{1}{2}}\tau$, for various values of magnetic interaction parameter $\alpha$.

magnetic diffusion regions. Only the dominant order one phenomena (namely, spin-up and Ekman–Hartmann layers) are certain to be given correctly by our approximation procedure; a much more subtle theory, correct to order $E^{\frac{1}{2}}$ or $\delta^{\frac{1}{2}}$ is required to elucidate the details of the weak velocities induced in these magnetic diffusion regions.

On the other hand, it is a simple matter to obtain the leading term in the radial velocity field outside of both Ekman–Hartmann layers and magnetic diffusion regions. The flow there is basically inviscid and current-free, so, by (13),

$$2U \doteq -V_\tau.$$

Together with (54) this implies

$$U(\zeta,\tau) \doteq -\tfrac{1}{2}\beta E^{\frac{1}{2}}\exp(-\beta E^{\frac{1}{2}}\tau), \tag{56}$$

and shows that $U$ obeys the Taylor–Proudman constraint to leading order, in a current-free core. The Ekman suction velocity accompanying this radial motion is (from (17))

$$W(\zeta,\tau) \doteq \beta E^{\frac{1}{2}}\zeta \exp(-\beta E^{\frac{1}{2}}\tau). \quad (57)$$

These last two results are hydromagnetic extensions of the approximate solutions found by Greenspan & Howard.

## 6. Summary and conclusions

Approximate inversion of an exact Laplace transform solution has shown that the dominant linear spin-up of an incompressible, electrically conducting fluid confined between infinite, flat insulating plates, is describable (in terms of dimensional quantities) by

$$\mathbf{v}(r,z,t).\hat{\theta} = r\Omega\{1+\epsilon[1-\exp[-\beta(\nu\Omega/d^2)^{\frac{1}{2}}t]]\}, \quad (58)$$

where $\beta = [\alpha^2+(1+\alpha^4)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad \alpha = (\sigma B_0^2/2\rho\Omega)^{\frac{1}{2}}$,

and $B_0$ is the imposed axial magnetic field. This formula for spin-up holds for the fluid outside of Ekman–Hartmann boundary layers and is valid (in an asymptotic sense) for small Ekman number $E$, small magnetic Prandtl number $\delta$, and for values of magnetic interaction parameter $\alpha$ of order 1 or less. (The function in (58), which is multiplied by the small parameter $\epsilon$, is plotted in figure 3.)

The spin-up time is always shorter than in the non-magnetic problem. However, the form of $\beta(\alpha)$ is such that a weak magnetic field is rather ineffective in promoting spin-up. For the larger values of $\alpha$ within the range of validity, $\beta \sim \sqrt{2}\,\alpha$; so the dimensional spin-up time is then

$$t_s \sim (d^2\rho/\nu\sigma B_0^2)^{\frac{1}{2}}, \quad (59)$$

which is independent of angular velocity, and increases as the first power of density (in contrast with non-conducting spin-up). It is perhaps surprising that for even fairly strong fields, doubling the field strength only halves the spin-up time.

Although the time required for the fluid to effectively reach the new angular velocity is basically independent of the thickness of magnetic diffusion regions, the mechanism of spin-up is strongly affected by these layers. Their dimensional thickness being $2(\lambda t)^{\frac{1}{2}}$, where $\lambda = 1/\mu\sigma$ is the resistivity, they grow to a fraction of the total fluid depth during the spin-up given by $\beta^{-\frac{1}{2}}E^{\frac{1}{4}}\delta^{-\frac{1}{2}}$, which can take on a wide range of values. When the layers remain thin throughout spin-up, they cause the Ekman suction velocity outside of them to be amplified somewhat over the classical value; spin-up of the essentially inviscid, current-free core then occurs by secondary flow, vortex stretching, and conservation of angular momentum (figure 1). If, on the other hand, magnetic diffusion is sufficiently rapid that electric currents permeate the entire fluid during spin-up, then it is accomplished by a weakened version of non-magnetic spin-up, augmented by an accelerating tangential body torque of hydromagnetic origin; specifically, a radially inward perturbation electric current interacts with the impressed axial magnetic field to produce an azimuthal component of the $\mathbf{j}\times\mathbf{B}$ force (figure 2).

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